# CLAUSEWITZIAN BLOTTO GAME

## A GAME-THEORETIC MODEL FOR THE ART OF WAR

by

YIREN ZHANG

(Under the Direction of Adam Goodie and O. Bradley Bassler)

ABSTRACT

Colonel Blotto games have been used as the model in strategic analysis on resource allocation since the 1950's. This paper implements an algorithm consistent with operational and tactical theories of Carl von Clausewitz in a probabilistic Colonel Blotto game. It transfers the empirical knowledge proven in history to a mathematical framework, creating new ways to understand strategic decision problems computationally and philosophically.

INDEX WORDS: [Colonel Blotto Game, Strategy, Game Theory, Resource Allocation, Operations Research]

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## DEDICATION

I dedicate this work to the bright-eyed Athena, the Inspirer of Humanity, mistress of the crafts of peace and war. Her name is the city of my residence.

And to my friends who philosophized and hiked with me, as well as those at Ramsey Aquatics, who, in their own ways, provided with me the respites on this journey.

Finally, and not the least of all, to my father, whose unwavering confidence and support sustained me in this endeavor.

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# CHAPTER 1

# BACKGROUND

## § 1.1 Introduction

### § 1.1.1 The Battle of Leuctra

The men of the Boeotian League arrayed themselves on the plain of Leuctra, in the summer of 371 BC, before the Lacedaemonian host. These men were led by the audacious Epaminondas, Strategos of Thebes, whose ambition to secure the League's future invited the enmity of Lacedaemon. An army 11,000 strong [15, Chapter 20] invaded Boeotia, marching behind the mighty Cleombrotus, king of Sparta. All that stood in their way, were the 7,000 Boeotians. [1][13, Chapter 9]. It was the triumph at Leuctra that propelled Thebes into hegemony, and secured Epaminondas' reputation as one of the most celebrated captains in antiquity.

This paper will use a variation of the colonel Blotto game to model this conflict, and analyze the tactics within the game theoretical framework. It is also an attempt to model the classical theories of conflict first promulgated in the 19th century by theoreticians like Carl von Clausewitz and Antoine-Henri Jomini. The insights from this analysis should not be interpreted as solely a matter of military history, but also a way of understanding decision making and resource allocation common in the field of operations research.

Although the connection between Clausewitz and game theory had been noted by [21][14], and with specific reference to Colonel Blotto Games in [9], the works have largely rely on concepts of strategic theories without modeling this understanding into an operational mechanism. This paper is an attempt to model a problem that could be solved using the operational concepts in Clausewitz' work.

## § 1.2 Colonel Blotto Game

Colonel Blotto Game was first proposed in Borel's paper in 1921 [2]. It is a two-player, zero-sum game with the following setup:

- Each player has $t$ amount of "troops" as the budget;
- The game is divided into $m$ areas of contests, or "battlefields", to which a non-negative amount of troops are assigned, such that $t > m^2$;

[1] Different historical sources provided conflicting numbers for the sizes of the armies. This paper uses Plutarch's in *The Life of Peliooidas* Chapter XX., which is considered to be a realistic estimation by Lazenby. The translation provided the next page reads "two thousand men-at-arms and one thousand horse". However, the Ancient Greek text is "ὁπλίτας μυρίους, ἱππεῖς δὲ χιλίους", which may suggest an error in the translation, as a μυρίους(myriad) and a χιλίους(chiliad) adds up to 11,000.

[2] The Blotto game is intended to have enough combinations of troops in battlefields to make the strategy problems interesting, but a Blotto game does not especially rely on $t$ outnumbering $m$ by several orders of magnitude to work. The diagrams in this paper uses a relatively small $t$ for visual clarity, but the strategies of a Blotto game does not especially rely on $t$ being orders of magnitudes greater than $m$ to work. For example, $m = 3, t = 4$ is a game with little depth in strategy, $m = 3, t = 10$

- Each player has $t$ amount of troops as the budget;
- For each battlefield, a Contest Success Function (will be abbreviated as CSF) determines the outcome of the contest;
- The game ends after the relevant conditions are satisfied.

Since [5] first formulated Borel's model as a military strategy simulation, the game has developed many variants. The game can differ chiefly in three areas: the CSF, victory condition, and manner in which the conflicts are resolved. Though the original Blotto game resolves all battlefields simultaneously, recent papers have developed a sequential variety, which assigns troops to the subsequent battlefield after the previous contest had been resolved, such as the gladiator game[7][16] This paper is draws inspiration from the time-sensitive nature of a sequential Blotto game, but instead uses time as a discriminator to shape the battlefields in contest. The troop assignment and resolution are effectively simultaneous.

### § 1.2.1 Model

If we can imagine two Greek phalanxes facing each other, we might get something similar to Figure 1.1: two tightly packed bodies of men that share similar qualities in most aspects, but differ chiefly in numbers, which is the factor modeled in Blotto games. The game can be formally defined as a series of battlefields $\mathfrak{B} = \{\mathfrak{b}^1, ..., \mathfrak{b}^m\}$. Each battlefield $\mathfrak{b}^i$ contains a pair of tuples$(s_A, s_B)$, where $s = (\mu, \kappa)$. $\mu$ represents the amount of troops assigned to the battlefield, and $\kappa \in \{-1, 1\}$ the form of action. The significance of each components will be discussed in the next chapter.

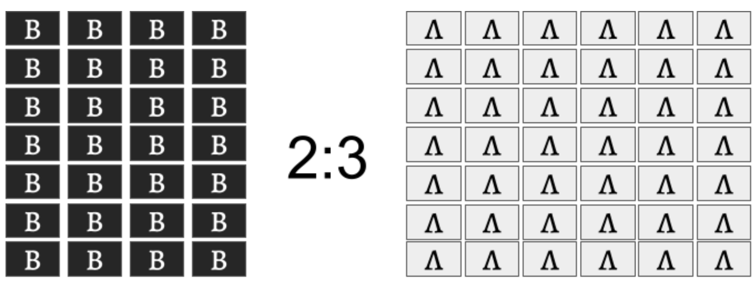



**Figure 1.1:** The balance of force

Figure 1.1 represents a strategic intuition that the Boeotians are not in a position of strength: If each row represents a Blottonian battlefield, we can draw the conclusion that for each battlefield, the Lacedaemonians on average can afford to put down 6 troops, whereas the Boeotians can only offer 4 troops. If $\mu$ is the only factor in deciding the outcome of this game, then the Boeotians are at a disadvantage. The strategy that can overcome this disadvantage to a large degree will be discussed in Chapter 3, and the mechanisms that supports this strategy will be explained below and in Chapter 2.

## § 1.3 Resolution

Before we explain how CSF resolve the conflict, attentions must be directed to the number of battlefields: Unlike the classical majoritarian Blotto game [10], the players are given their victory objects

and $m = 50, 547, 438$ have different sizes of problem space, but both have the strategic depth to be interesting. This paper assumes $t$ to be sufficiently large.

exogenously as $w$'s, whose sum is smaller than the total number of battlefields. Therefore there is no need to continue the game after $|\mathfrak{C}|$ number of battlefields are resolved, because after so many resolutions, it is guaranteed that one side would have achieved $w$ wins. We will use $\mathfrak{C}$ and $\mathfrak{D}$ to represent the set of battlefields that are in the contest, and the set of those that serve only as diversion For this paper, we will assume that $|\mathfrak{D}| > |\mathfrak{C}|$, because the effects of concentration, which will be discussed in the following chapters, is amplified as $|\mathfrak{D}|$ increases. The resolution has two parts: firstly, to determine $|\mathfrak{C}|$, and secondly, to find the winner of the $|\mathfrak{C}|$-battlefield contest, both will use the Tullock CSF.

The Tullock lottery function [22] is a widely studied stochastic CSF. It derives the probability of success for Player A on the battlefield $i$ based on the $\mu_A^i$ and $\mu_B^i$:

$$p_A^i = \frac{\mu_A^i}{\mu_A^i + \mu_B^i} \tag{1.1}$$

It is a general model that represents the degree of control a player has over an adversarial situation given the player and the opponent's investment of troop resources - the more troops by which the player outnumbers the opponent, the more likely the what the player intends will come to pass.If the player wishes the battlefield to be contested, then $\kappa = +1$ should be assigned, and $-1$ if otherwise, The player may still have to assign troops to the battlefields that they do not wish to be entered into the contest because a negative $\kappa$ allows the troops to slow down the resolution priority, which will be discussed in detail in Chapter 2. We can put all battlefields on a timeline based on $\chi$ in descending order, where:

$$\chi^i = p_A^i * \kappa_A^i + p_B^i * \kappa_B^i \tag{1.2}$$

Battlefields arranged in $\chi$-descending order represents the timeline of the battle: The battlefields with higher $\chi$ is promoted in the order. Equation 1.2 does not necessarily represent a contest, because when both players have the same $\kappa$ signs, they are in agreement, which would put them in the first or the last on the timeline. When a battlefield has $\chi = -1$, it will not be resolved. Therefore, $p_A$ serves double duty in this game: to take the battlefield and to move the resolution earlier or later in the timeline when the two players disagree on their $\kappa$ values.

# CHAPTER 2

## FORMS AND OBJECTIVES

For this to be termed a Clausewitzian game, its mechanism will be explained in his theory of strategy. It is not difficult to understand the role of $\mu$ in a strategy, as they are present in all Blotto games. But $\kappa$ is a novel addition: The Clausewitizian conception of conflict begins with the forms, based on their effects, he divided actions into those with a positive objective and the those with the negative objective. The positive and the negative are often associated with the attack and defence respectively. This biform framework is distinct from a uniform, zero sum game, in which a player can only attack, and loses exactly as much as the other side gains. Such perfect polarity is closer to fiction than to real life[3] (Book I, Sections 14-17), whose complexities often contains mitigating factors.

**Definition 1.** A positive objective is the objective of progress. A negative objective is that of deferral.

Conceptually, when a player continues to achieve positive objectives, they would eventually win. If they continues to achieve negative objectives, against the opponent's positively motivated actions, then their opponent cannot win. A battlefield does not have a contest if no player has a positive objective on it – The player must achieve positive objectives to win the game, and a mutually defensive battlefield does not contribute to this overarching process, which is why they will never be in the contest.

However, it is better not to commit to an unfavorable decision, but to wait until a more opportune moment to arise (Book I, Chapter 1, Section 13).This offence-defence dichotomy decouples the circumstances on one side from the other side when analyzing $\mu$ and $\chi$, making the analysis reflexive. In a game where attack is the only action, if A has a disadvantage on a battlefield to attack B, translates to B's advantage to attack A. But Clausewitz observed that in war, the only conclusions that can be drawn safely is that B has an advantage to be attacked by A( in this case, A is attacking, and B defending). It is a statement solely about A's decision, and does not translate to B should attack as before. Because attack and defence are two different games, it is harder to attack than to defend. In Book VI Chapter 1, Clausewitz remarked that defence is inherently the stronger form, meaning on the same battlefield, the side with the negative objective can achieve it with less troops than the side with positive objective. As is with everything governed by inertia, it is easier to keep the status quo. When it is disadvantageous to attack or defend, one can either adjust troop numbers or change to the other form.

The function of positive objectives is two fold: mechanically, to take the required $w$ battlefields. In order to achieve decisive victories however, it must also prevent the opponent from doing the same. Both [6] (Supplementary) and Clausewitz (Book IV, Chapter 9) identify the neutralization of the opponent's forces as the main objective in a war – one loses when one has lost the army. If both sides loses their armies, then the situation remains indecisive. Therefore, the number of positive objectives is bounded by the constraint. In effect, its number shapes the size of contest $|\mathfrak{C}|$.

It should be noted, therefore, that having attained the required number of battlefield wins is not automatically winning the game: In the beginning of the game, neither player attains the required $w$ battlefield wins to win the game, this situation is an indecisive equilibrium. As we resolve more and more battlefields, at some point we will have a situation where one player has collected enough battlefield wins, but the other has not; At this point, the equilibrium is upset, and the result decisive. If the battlefield resolutions continues however, we may face another kind of indecisive equilibrium in which both sides have reached their numbers. If to win in a duel is to destroy one's opponent, then a "win-win" scenario is one in which both sides are destroyed.

**Definition 2** (Mutual Destruction)**.** If both players achieved all of their respective positive objectives, then neither can claim victory.

If the positive objective means to upset the equilibrium, then it must increase to prevent the equilibrium from restoring itself, which means higher cost of resources. Mutual destruction is a concern when considering the scope of the contest $|\mathfrak{C}|$, because the probability of mutual destruction increases if the scope is too large.

### § 2.0.1 Overall Chance

There are ${}_nC_h$ ways for Player A to win $h$ out of $n$ battlefields. Let $N = \{1, ..., n\}$, the set $F_h$ be the enumeration of all the subsets $S \subset N$ such that $|S| = h$, which represents the battlefields won in the scenario. We use $S^c$ to represent the complement of $S$, which is the battlefields lost in the same scenario. Then the probability of winning is the joint probability $\prod_{j \in S} p_A^j \prod_{h \in S^c} p_B^j$. For A to win against the belief of B's strategy $f_B$', he must win least $w_A$ out of $|\mathfrak{C}|$ battles. and at most $|\mathfrak{C}|$, the probability of winning the game is :

$$P_A(f_A, f'_B) = \sum_{h=w_A}^{n} \sum_{S \in F_h} \prod_{j \in S} p_A^j \prod_{h \in S^c} p_B^j \tag{2.1}$$

The calculation in Equation 2.1 is simplified to a cumulative binomial probability function in a special case when we know $p_A$ is the same value for all battlefields[8]:

$$P_A(f_A, f'_B) = \sum_{i=w_A}^{n} {}_nC_i * {p_a}^i * (1 - p_a)^{(n-i)} \tag{2.2}$$

A further special sub-case of binomial probability when the the number of required successes is the same as the number of trials, it becomes the joint probability:

$$P_A(f_A, f'_B) = {p_A}^{w_A} \tag{2.3}$$

We can compare $P_A$'s from different strategies to evaluate the effectiveness of each strategy.

### § 2.0.2 Operational Implications of the Forms

The outlook of a game would be different with the same troop assignment, depending on whether they are pursuing a negative or positive objective. The game strategy can be divided into two groups: Contest ($\mathfrak{C}$), which contains battlefields that will be resolved; Diversion ($\mathfrak{D}$), which includes the rest of the battlefields. The contest includes the battlefields with the highest $\chi$ values until the $(w_A + w_B - 1)$th battlefield is reached. Because all battlefields with the same $\chi$ occupies the same spot on the timeline, any additional battlefields that share the $\chi$ value would also be included, in which case the contest is extended. The number of battlefields in the contest $|\mathfrak{C}|$ does not mean how long the actual game would last, but as a maximum possible length it defines a parameter of the strategy. A strategy to win at least three battlefields out of five is necessarily different from a strategy to win at least three battlefields out of a million.

When $|\mathfrak{C}|$ is larger than necessary, it is possible for both players to achieve their respective objectives, in which case Definition 2 will apply, and the winner must win more battlefields to prevent mutual destruction.

$$w_A = \begin{cases} w_A & n \leq w_A + w_B - 1 \\ n - w_B + 1 & n > w_A + w_B - 1 \end{cases} \tag{2.4}$$

A good strategist should avoid such expansive contests, for they make winning the game harder for either player regardless of their strategies within these contests.

# CHAPTER 3

# GENERAL METHOD

Our goal is to upset the indecisive equilibrium in A's favor by coming up with a better strategy. This chapter will provide the algorithm, and discuss the game mechanisms it exploited.

## § 3.1 Algorithm

The previous chapter has illustrated the effect of disposition and posture of troops in a battle, which shall be the considerations of strategy discussed in this chapter. The core idea of strategy was to cherry pick the battlefields in which the player outnumbers the opponent for contest. By excluding enough of the opponent's troops, one can upset the equilibrium.

The primary interest of this paper is to establish a framework: Because we take $f_B$' to be true, we only have to deal with pure strategy. The following functions are used to find the battlefields that require attention:

$$\begin{aligned} c+ &= index(\arg\max_{b \in \mathfrak{C}} \chi(b)), & c- &= index(\arg\min_{b \in \mathfrak{C}} \chi(b)), \\ q+ &= index(\arg\max_{b \in \mathfrak{C}} p_A(b)), & q- &= index(\arg\min_{b \in \mathfrak{C}} p_A(b)), \\ d+ &= index(\arg\max_{b \in \mathfrak{D}} \chi(b)), & d- &= index(\arg\min_{b \in \mathfrak{D}} \chi(b)) \end{aligned}$$

*c+, c-, d+, d-* locate the battlefield of with the greatest/least $\chi$ values in the contest and diversion respectively. While $q+$, $q-$ locate the battlefields with the greatest/least $p_A$ values in the contest. The algorithm change the troop disposition through adjustments on these battlefields, see Appendix for the an example that shows troop disposition at each major step.

**Algorithm 1:** Troop Allocation in Symmetrical Contests

**Input:** $f_B{}'$
**Output:** $f_A$
Arrange $f_B{}'$ in $\mu$-ascending order;
**if** $s_B{}^x$ *and* $s_B{}^y$ *have the same* $\mu$ *value* **then**
    The one with higher $\kappa$ takes precedence ;
**end if**
let $\widehat{\mu_A} = \frac{t_A}{m}$;
Initialize strategy $f_A = \{(\widehat{\mu_A}, +1)^1, ..., (\widehat{\mu_A}, +1)^m\}$ , and reserve $R = 0$;
Let $|\mathfrak{C}| = w_A + w_B - 1$, set $\kappa_A^{n+1}, ..., \kappa_A^m$ to $-1$;
Let $\mathfrak{C} = \{\mathfrak{b}^1, ..., \mathfrak{b}^n\}$ and $\mathfrak{D} = \{\mathfrak{b}^{n+1}, ..., \mathfrak{b}^m\}$;
**while** $\chi(\mathfrak{b}^{d-}) = -1$ **do**
    Transfer all troops on that battlefield to $R$;
**end while**
**for** $i \in [|\mathfrak{C}| + 1, m]$ **do**
    **if** $\chi^i > -1$ **then**
        Transfer troops from $s_A^i$ to $R$, such that $\chi^i$ is as close to $\chi^{d+}$ as possible without exceeding it;
    **end if**
**end for**
**while** $\chi^{c-} \leq \chi^{d+}$ **do**
    Transfer troops from $R$ to $\mu_A^{c-}$;
**end while**
**while** $R > 0$ **do**
    **if** $\chi^{c-} \leq \chi^{d+}$ **then**
        Transfer troops from $R$ to $\mu_A^{c-}$;
    **else**
        Transfer troops from $R$ to $\mu_A^{q-}$;
    **end if**
**end while**
**return** $f_A$

This type of problem, in which we try to find and invest in the battlefield that gives the player most payoffs, may be generalized as a reinforcement learning problems, like the multi-armed bandit [20][1]. The agent may be trained to gravitate towards battlefields with small $\mu_B$ values, so as to increase the probability of success. But the game $\mathfrak{B}$ can be seen as a metagame to the game $\mathfrak{C}$ which is the *real* game because it guarantees decisive outcomes, and its outcome determines the outcome of the game $\mathfrak{B}$. $\mathfrak{C}$ is not played with the starting $t_A, t_B$ or $m$, but with the troops of both sides that were assigned to the first $(w_A+w_B-1)$ battlefields on the timeline, and $\mathfrak{C}$ is not defined at the beginning of the game, but is shaped by the strategies of the two sides. Therefore our strategy has two purposes: first, to configure the contest battlefield, and second, to optimize probability of success of this contest. The strategist (agent) is not solely motivated by the improvement of probabilities of success, but is also constrained by the chronological priority. This

[1]Considering that there are ${}_mC_{(w_A+w_B)}$ combination of battlefields, it may be a substantial number.

algorithm, and *Schwerpunkt* as the concept itself, should be viewed as heuristics that would save time and computational resources when dealing with such problems. Once $\mathfrak{C}$ is defined and $R$ is determined, the rest of the game may be solved with machine learning as usual. The main contribution to our strategic concept is the shaping $\mathfrak{C}$, rather than playing out the contest.

The purpose of the algorithm is to find the battlefields in which B is weak, to create advantages in these battlefields, and finally to ensure that these battlefields are resolved. Our understanding of strength is both quantitative and qualitative: when $\mu_B^i < \mu_j$, then B is weaker in $\mathfrak{b}^i$ is than in $\mathfrak{b}^j$. If $\mu_B^i = \mu_j$, and $\kappa^i = -1, \kappa^j = 1$, then then B is stronger in $\mathfrak{b}^i$, because $\kappa^i = -1$ is the stronger form.

## § 3.2 The *Schwerpunktprinzip*

From the example in the foregoing chapter, we can see that the contest in $\mathfrak{C}$ battlefields, when $|\mathfrak{C}| = w_A + w_B - 1$, ensures that one side achieves all of its positive objectives, and the other side does not. This phenomenon encompasses the notion of "decisiveness".

**Definition 3** (Decisive Contest)**.** A contest between two players is decisive if and only if there can only be one winner and one loser from the contest.

Operationally, only $|\mathfrak{C}| = w_A + w_B - 1$ guarantees this outcome.

Our intuition should tell us that the probability of winning is related to how many troops are in these battlefields. $\mathfrak{C}$ is identified, in the Prussian tradition of strategic thinking, with the concept of *Schwerpunkt*("Center of Gravity").

**Definition 4** (*Schwerpunkt* as a strategic concept)**.** A *Schwerpunkt* is the source of counteracting force that maintains the equilibrium.

The upsetting of the equilibrium of force, by the means of neutralizing the source of the opponent's ability to resist, will end the contest in one's favor, thus was a decisive victory achieved. The Prussian war philosopher wrote:

> The skilful assemblage of superior forces at the decisive point—has its foundation in the right appreciation of those points, in the judicious direction which by that means has been given to the forces from the very first, and in the resolution required to sacrifice the unimportant to the advantage of the important — that is, to keep the forces concentrated in an overpowering mass.(Book III, Chapter 8)

It follows, therefore, that all non-decisive battlefields must be ruthlessly sacrificed in order to achieve such concentration.

In order to operationalize the *Schwerpunkt* concept, it is necessary to show how it fits in game mechanics: The equilibrium is maintained until one side satisfies its $w$ condition. Therefore the strategic objective translates to the operational imperative to win $w$ battlefields before the opponent, specifically the $w$ battlefields in which the opponent is least able to overcome. The weak points are those with the least amount of troops. When the numbers are the same, the troops that are attacking weaker than those in defence, since attack is the weaker form of the two.

We will begin with an example to illustrate the effect of changing each variable in the strategy.

Notations and legends were introduced for the ease of description and presentations, which are explained as follows:

1. On diagrams, + indicates that the entire file of the same color has the positive objective, - likewise indicates a negative objective;
2. Unless specified in subscripts, the values for $t$ and $w$ are for both players;
3. Strategies are written in a simplified notation, e.g., $f_A = \{(5, +1), (5, -1)\}$ is written as $f_A = \{+5, -5\}$;
4. When describing the victory condition for the contest, we use $w_x/|\mathfrak{C}|$ to represent the notion "x must win $w_x$ battlefields out of $|\mathfrak{C}|$ to win the game". e.g., "A must win at least 3 out of 5 battlefields to win the game" is expressed as 3+/5;

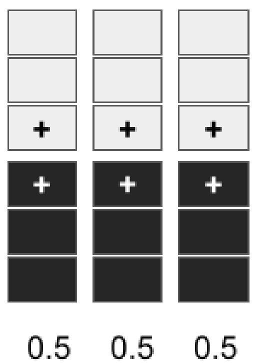



**Figure 3.1**

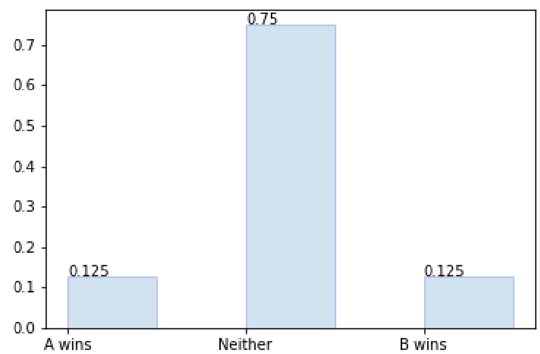



**Figure 3.2**

Assuming B does not change, A can move the troops from one battlefield to another to adjust the $\mu$ values. But admittedly there is no point in such manoeuvre, because the strategy of spreading troops evenly is the Nash equilibrium, in this unfortunate case however, $P_A = P_B = 0.125$. Any unilateral action from A will upset this equilibrium in the opponent's favor.This would have been a sound strategy without the mutual destruction rule. This would normally be the end of game theory analysis, but the comparison of Figures 3.2 with 3.4 reveals a strange picture: The players are not only at odds with each other, but also with the mutual destruction rule. Any deviation from the optimal strategy increases the chance of mutual destruction. On the whole, the game is not worth playing.

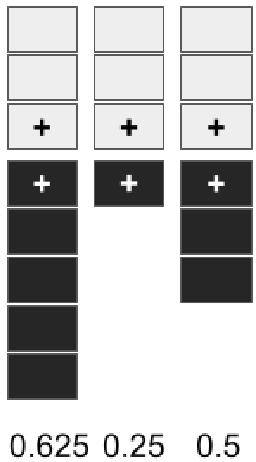



**Figure 3.3**

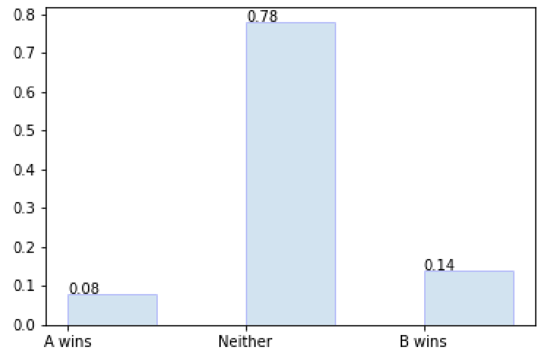



**Figure 3.4**

The probability of mutual destruction comes from the extra battlefields in the contest. If the game requires each player to win at least 2 games, A is still worse off concentrating troops than splitting them evenly.

## § 3.3 The Oblique Order

The only conclusion we can draw from the previous example must be a refutation of the *Schwerpunkt* principle. A complete concentration of all available troops on the decisive contest is impractical, because it leaves all the other battlefields undefended. Any moderate attempt at concentration merely takes a smaller step in the same wrong direction: To deviate from the evenly-split strategy is to walk away from the Nash equilibrium, hence the paradox: One can only achieve a local advantage by unilateral concentration, but if the effect of the weakening in other battlefields comes at the same time, the cost of a global disadvantage is greater than the benefit of concentration. Optimal mediocrity, it seems, prevails in such games.

Hitherto we have only discussed the probabilistic considerations with respect to troop division. We know that positioning troops merely trade a probable victory in one battlefield for a more probable disaster in another. The temporal dimension gives the players agency to pick the battles they want to fight through a negotiation that creates the opportune moment for the decisive action.

Let us now consider the formation of troops, for now we specifically mean bodies of soldiers, rather than the resources in the model. There are two basic formations widely used throughout history: that of the parallel order and the oblique order[6] (Chapter 4, p.147). If we consider a battlefield as a 2-dimensional space, everyone in a formation parallel to the opponent's would meet their opposite at the same time. By comparing these distances of two sections, we can find out which section concludes its fighting first. In a parallel order, every fight begins at the same time. Jomini regarded the parallel order as almost always undesirable, and inferior to the oblique order. The latter is achieved by refusing one wing[2] and advancing the other: The formation deployed in such a fashion resembles a diagonal line, allowing the advanced wing to make contact before the center, and the center before the refused wing. The following diagrams illustrates the difference:

[2] It means this side of the formation is held back.

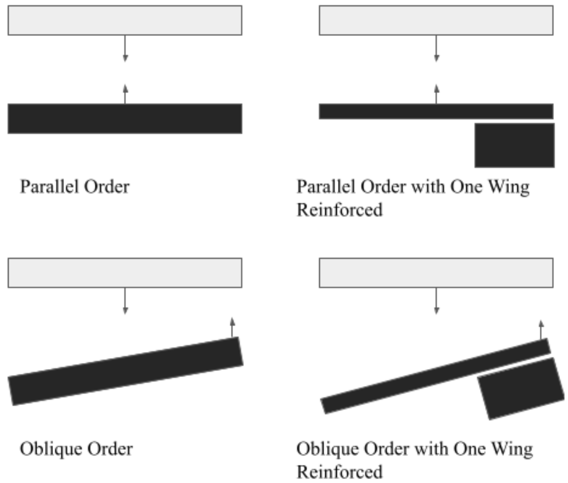



**Figure 3.5**

The oblique order breaks up one large event into a sequence of smaller events. The advantages it confers corresponds to the advanced and refuse wings: the advanced wing pushes its contests ahead on the timeline, naturally suited for positive objectives. The refused wing has the same property as a negative objective, whose aim is to delay the decision. The synergy between the oblique order and the *Schwerpunkt* is unmistakable: It allows the tactician to shift a considerable mass of troops to the advanced wing, without *simultaneously* suffer the penalty of weakening the rest of the formation, creating the opportune moment for the decisive action.

In summary, the solution to the paradox contains two parts: Oblique order adds a temporal dimension to the problem space, which can be qualitatively divided into the "before" and the "after". This extra degree of freedom allows concentration of troops in the "before" part of the game to improve the chance without repercussions.

The effect of excising the temporal freedom is demonstrated below: Still assuming $w = 1andm = 3$, the only course is to reduce the size of the contest $|\mathfrak{C}|$ instead. We partition the game by using defense to delay some of the battlefields, the game is no longer a singular event that encompasses all outcomes at once, but spreading them over a period of time. For example:

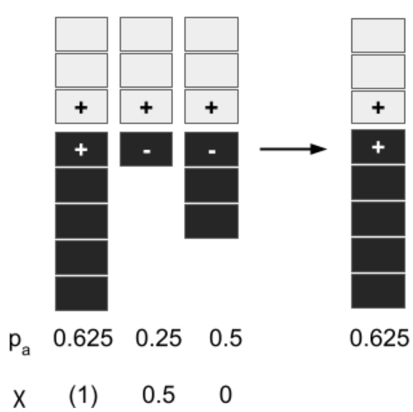



**Figure 3.6**

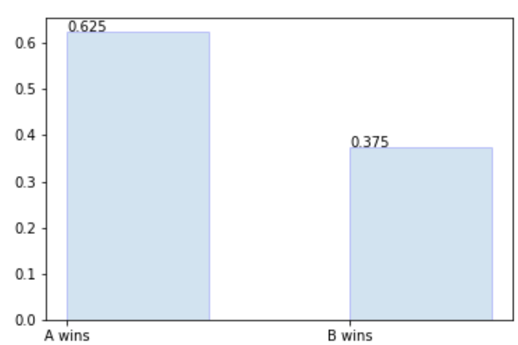



**Figure 3.7:** Winning a 1/1 game with $p_A = 0.625$

By adopting a defensive posture at $\mathfrak{b}^{2,3}$, A is able to reduce the scope of the contest to only $\mathfrak{b}^1$, in which A enjoys a relative superiority. Since the previous probability of success was 0.08, which has turned into 0.625, we can conclude that the strategy $\mathfrak{f}_A = \{(+5, -1, -3\}$ is superior to $\mathfrak{f}_A = \{+5, +1, +3\}$ against what A believes to be B's strategy $\mathfrak{f}'_B = \{+3, +3, +3\}$ A does not improve the chance by clever disposition of troops, but simply by controlling the agenda of the battlefield resolution. It is not hard to imagine $t_A$ being less than $t_B$, and still managed to achieve a disproportionately high chance of success in this way.

The probability of winning the game $P_A$ is affected by several factors:

- The strategy of the opponent $\mathfrak{f}_B{}'$;
  - The number of troops $\mu_A, \mu_B$ in each contested battlefield;
  - The number of battlefields in contest $|\mathfrak{C}|$;
- The total number of battlefields; $|\mathfrak{B}|$
- The number of total troops $t_A, t_B$;
- The number of positive objects $w_A, w_B$;

Our focus is on games that are more or less symmetrical, here we mean any game with $t_A \simeq t_B, w_A = w_B$. Traditional definition of symmetry, in a majoritarian Blotto game, requires $t_A = t_B$, which [17] has explored. But since part of our motivation is to show how the side with less troops could still create a strategy that dominates the opponent's, this requirement is relaxed. Generality of the observations is not lost when $w_A \neq w_B$, however. The difference in troop numbers may be great in some circumstances due to the strategies chosen by both sides, but in all cases, $t$ must be sufficient to allow the player to complete the same number of objectives as their opponent does.

## § 3.4 Contest and Diversion

The key to the success of a *Schwerpunkt* is to separate the troops at the resolution of the $|\mathfrak{C}|$-th battlefield. In another word, we would like to show that after concentration, $\chi^{c-} > \chi^{d+}$. We restrict this analysis to the scenarios when $\chi^x = p_A - p_B$. Therefore, we can safely say that if the battlefield $\mathfrak{b}^x \in \mathfrak{C}$, $\mathfrak{b}^y \in \mathfrak{D}$ and $\chi^x > \chi^y$, then $p_A^x > p_A^y$. In another word, when A manages to move a battlefield into the contest phase despite opposition, then this A is more likely to win in this battlefield than the ones in the diversion.

**Lemma 1.** Given $\mu_B^i < \mu_B^j$, if $\mu_A^i = \mu_A^j$, then $p_A^i > p_A^j$.

*Proof.* $p_A^i > p_A^j = \frac{\mu_A^i}{\mu_A^i+\mu_B^i} > \frac{\mu_A^j}{\mu_A^j+\mu_B^j}$. Since $\mu_A^i = \mu_A^j$, we will use $\mu_A$, then

$$\begin{aligned}\frac{\mu_A}{\mu_A+\mu_B^i} > \frac{\mu_A}{\mu_A+\mu_B^j} &= \mu_A(\mu_A+\mu_B^j) > \mu_A(\mu_A+\mu_B^i)\\ &= \mu_A{}^2 + \mu_A\mu_B^j > \mu_a{}^2 + \mu_A\mu_B^i\\ &= \mu_A\mu_B^j > \mu_A\mu_B^i\\ &= \mu_B^j > \mu_B^i\end{aligned}$$

□

Lemma 1 tells us where to attack: with the same amount of investment $\mu_A$, A is more likely to succeed in $\mathfrak{b}^i$ than in $\mathfrak{b}^j$. This is due to the denominator in the equation. Therefore we can predict that the magnitude of change is greater in $\mathfrak{b}^i$ , given the change in $\mu_A$ in both battlefields.

If $\mu_B^i < \mu_B^j$ and $\mu_A^i = \mu_A^j = \mu_A$, then for any non-zero value $r$, the differences of add $r$ in $\mathfrak{b}^i$ and $\mathfrak{b}^j$ are:

$$\left|\frac{r}{\mu_A + r + \mu_B^i}\right|, \text{ and } \left|\frac{r}{\mu_A + r + \mu_B^j}\right|$$

Note that operationally, $r$ would have to be an integer rounded toward zero[3].

**Corollary 1.1.** If $\mu_B^i < \mu_B^j$ and $\mu_A$ is constant, then for any non-zero value $r$,

$$\left|\frac{r}{\mu_A + r + \mu_B^i}\right| > \left|\frac{r}{\mu_A + r + \mu_B^j}\right|$$

A transfer of $r$ troops from $\mathfrak{b}^j$ to $\mathfrak{b}^i$ is good for improving $p_A$, if and only if

$$\left|\frac{r}{\mu_A + r + \mu_B^i}\right| > \left|\frac{-r}{\mu_A - r + \mu_B^j}\right|$$

From which we take the denominator

$$(\mu_A + r + \mu_B^i) < (\mu_A - r + \mu_B^j)$$

$$(\mu_B^i + r) < (\mu_B^j - r)$$

let $r$ be such value that

$$(\mu_B^i + r) = (\mu_B^j - r)$$

Since $\mu_B^j > \mu_B^i$, $0 < r < \frac{\mu_B^j - \mu_B^i}{2}$.

From Equation 1.2, we can rewrite $\chi$ as a function:

$$\chi^i = \begin{cases} p_A - p_B = \frac{\widehat{\mu_A}}{\widehat{\mu_A}+\mu_B^i} - (1 - \frac{\widehat{\mu_A}}{\widehat{\mu_A}+\mu_B^i}), & i \in [1, |\mathfrak{C}|] \\ -p_A + p_B = -\frac{\widehat{\mu_A}}{\widehat{\mu_A}+\mu_B^i} + (1 - \frac{\widehat{\mu_A}}{\widehat{\mu_A}+\mu_B^i}) & i \in [(|\mathfrak{C}| + 1), m] \end{cases} \tag{3.1}$$

for when A uses an evenly split strategy, and the $\kappa$'s of the two sides do not match on $\mathfrak{b}^i$. We know that in a decisive game, $p_B = 1 - p_A$, therefore

then the differences in $\chi$ from a transfer of $r$ troops from $\mathfrak{b}^i$ to $\mathfrak{b}^j$ are:

$$\chi^{i\prime} = \frac{\widehat{\mu_A} + r}{\widehat{\mu_A} + r + \mu_B^i} - (1 - \frac{\widehat{\mu_A} + r}{\widehat{\mu_A} + r + \mu_B^i}) \tag{3.2}$$

$$\chi^{j\prime} = (1 - \frac{\mu_B^j}{\widehat{\mu_A} - r + \mu_B^j}) - \frac{\mu_B^j}{\widehat{\mu_A} - r + \mu_B^j} \tag{3.3}$$

[3] $r < 0$ represents the amount of troops transferred in the reverse direction. To round toward zero is to round up if $r < 0$, and down if $r > 0$.

for $\mathfrak{b}^i \in \mathfrak{C}$ and $\mathfrak{b}^j \in \mathfrak{D}$. Because $\chi^i = p_A^i - (1 - p_A^i)$, the effect of $r$ additional troops is doubled from that of $p_A$. Let $\chi^{i\prime}$ be the value after receiving $r$ troops:

$$\begin{aligned}
\left|\chi^i - \chi^{i\prime}\right| &= \left|(p_A - (1 - p_A)) - ((p_A + \frac{r}{\mu_A + r + \mu_B^i}) - (1 - (p_A + \frac{r}{\mu_A + r + \mu_B^i})))\right| \\
&= \left|2 * p_A - 1 - 2p_A + 2 * \frac{r}{\mu_A + r + \mu_B^i}\right| \\
&= \left|2 * \frac{r}{\mu_A + r + \mu_B^i}\right|
\end{aligned}$$

and vice versa for $\chi^j$. This means that the same general pattern as shown in Lemma 1 holds true for both probability of success $p$ and chronological priority $\chi$: If holding $\mu_A$ constant, the battlefields with a higher $\mu_B$ are more stable than those with lower.

**Theorem 2.** If $\mu_B^1 < \mu_B^2 <, ..., < \mu_B^m, \mu_A^1 = \mu_A^2 =, ..., = \mu_A^m = \widehat{\mu_A}$ and $\widehat{\mu_B} = \widehat{\mu_A}$, then there exist a positive value $r$ in $\mu_B^{d-}$, such that

$$\chi^{d-\prime} \leq \chi^{d+} < \chi^{c-\prime}$$

*Proof.* Suppose $r$ is at such value that

$$\chi^{d-\prime} = \chi^{d+}$$

since $\mu_B^{c-} < \mu_B^{d-}$, $\chi^{c-}$ is improved to a greater extent than $\chi^{d-}$.

$$\left|2 * \frac{r}{\mu_A + r + \mu_B^{c-}}\right| > \left|2 * \frac{r}{\mu_A + r + \mu_B^{d-}}\right|$$

if $\chi^{d-\prime} = \chi^{d+}$, then $\chi^{c-\prime} > \chi^{d+}$. □

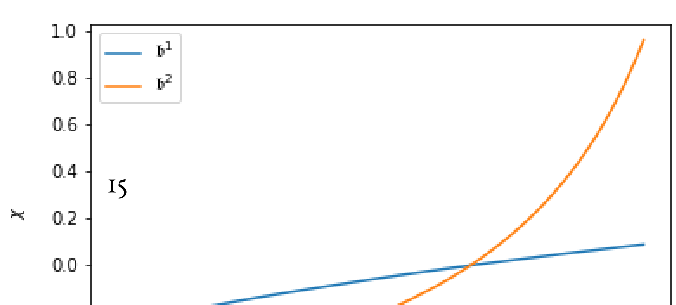
This would shows that concentric movement can be a profitable excise when pursued without excess. Not only will it improve $p$, but also prevent $\mathfrak{b}^{d+}$, or any other diversionary battlefields from being counted as a part the contest.

When quantitative advantages are not clear, such as in the event of facing an evenly split strategy, our troop disposition is much simpler: assign minimal troops to each diversionary battlefields, because the contest is attacking into an attack with $\chi^{c-} = 1$, a diversion is successful so long as the $\chi^{d+} < 1$.

## § 3.5 Troops: Transferal and Total Numbers

When the improvement in $\chi^i$ allows it of offset the change in $\chi^j$, A is able to control the timing. Interestingly, this ability is more affected by $f_B$ than $t_A$ and $t_B$. Suppose we have two battlefields $\mathfrak{b}^i \in \mathfrak{C}$ and $\mathfrak{b}^j \in \mathfrak{D}$. In either battlefield, $\widehat{\mu_A} = 150$, $\mu_B^1 = 100$, $\mu_B^2 = 200$. If we began to transfer troops $\mu_A^j$ to $\mu_A^i$, we can expect the marginal utility, i.e. the amount by which $\chi$ is changed per transferred troop, to decrease in $\mathfrak{b}^i$ and increase in $\mathfrak{b}^j$, until A must stop in order to avoid subverting the preferred temporal relation $\chi^1 > \chi^2$. This threshold looks like Figure 3.8:

This threshold is the maximum amount of troops A can spare from the diversionary battlefield without jeopardizing control of the timeline - the



maximum concentration that can be achieved in $\mathfrak{b}^1$. This number may be smaller when more battlefields are involved, since $\chi^{c-}$ is the threshold. We observe that $\mu_A$ can reach a strategic critical mass, sometimes below $\mu_B$, to achieve certain leverage. Nevertheless, the greater number, when used properly, is always an advantage. On the other hand, when this critical mass is not reached, A would not be able to formulate any meaningful concentration-based strategy, and is completely at the mercy of fortune. The critical mass of A is, to a large degree, affected by the B's own degree of concentration:

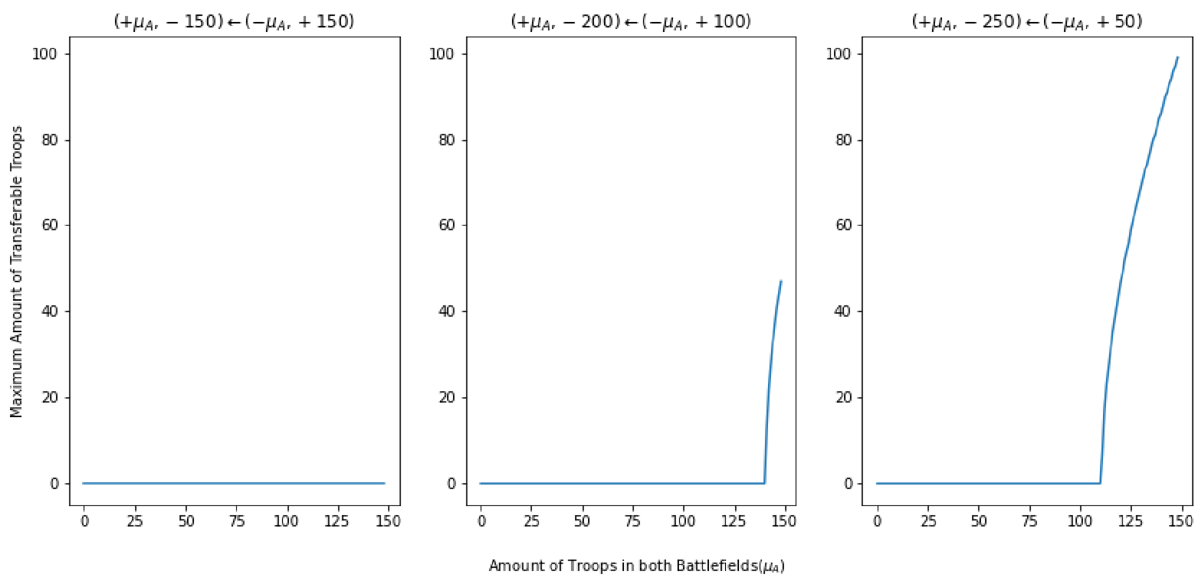



**Figure 3.9:** Depends on how B assigns troops and how many troops A has, the latter might be able to spare the majority of defensive troops or none at all.

The first graph describes an awkward situation where A cannot take off a single troop without attaining the overall numerical superiority themselves $\{(+150, -150), (-150, +150)\}$. The impasse can be circumvented by prioritizing battlefields that are also the opponent's positive objects $\{(-150, -150), (+150, +150)\}$. This consideration is not based on numerical weakness, but due to the troops of having the positive objectives, and therefore are weaker than those who have negative objectives. This situation makes the diversionary battlefields stronger than the contest, which means A could dominate B in the contest by concentrating to a greater degree on the weaker battlefields. The best strategy, noted Clausewitz: "is *always to be very strong*, first generally then at the decisive point. " (Book III, Chapter 11) [4]. Here, the strength on a battlefield should not be interpreted as troop numbers on each battlefield, but the likelihood of the troops succeeding their given objectives: A symmetrical game does not allow A to have $p_A > 0.5$

[4]emphasis is original.

on every battlefield. Therefore, the diversionary battlefields are not meant to be that strong, just strong enough to prevent the opponent from achieving their positive objectives in time.

If we look at the middle and the right graph, we see that A was able to spare troops when $\mu_A = 141$ and $\mu_A = 111$ respectively. If we only consider these two battlefields, it means A is capable of wresting away B's control of timeline at$t_A = 282$ and $t_A = 222$ respectively, even if B outnumbers A with $t_B = 300$. This advantage multiplies with number of battlefields. If there were two of each such battlefields, A would be able to attain the same strategic advantage with 564 and 444 troops against B's 600 troops, confirming the observation that the weaker player has relatively greater advantage when there are more battlefields.

# CHAPTER 4

## SIMULATIONS

The games below explore different scenarios in order to demonstrate the concepts outlined in Chapters 2 and 3. Notations and legends were introduced for the ease of description and presentations, which are explained as follows:

1. On diagrams,  indicates that the entire file of the same color has the positive objective,  likewise indicates a negative objective;

2. Unless specified in subscripts, the values for $t$ and $w$ are for both players;

3. Strategies are written in a simplified notation, e.g., $f_A = \{(5, +1), (5, -1)\}$ is written as $f_A = \{+5, -5\}$;

4. When describing the victory condition for the contest, we use $w_x/|\mathfrak{C}|$ to represent the notion "x must win $w_x$ battlefields out of $|\mathfrak{C}|$ to win the game". e.g., "A must win at least 3 out of 5 battlefields to win the game" is expressed as 3+/5;

### § 4.1 Attack and Defence

This example demonstrates the effect of defence with respect to the contest:

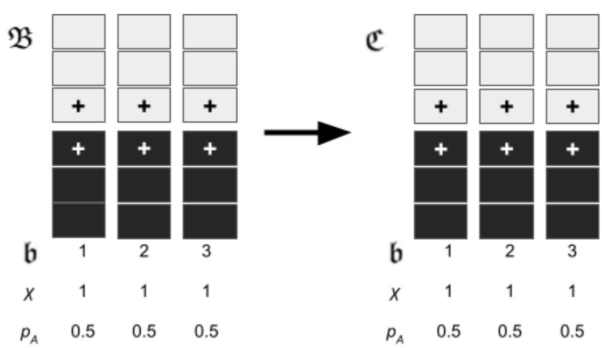



**Figure 4.1:** When both sides are attacking, the game is resolved in the shortest time

The extent of delay depends on the number of troops, but the effect of delay is present regardless of $\mu$, to achieve this qualitative result with minimum investment is a pillar of this strategy.

### § 4.2 *Schwerpunktabschnitt*

Vego [23] defines the term *Schwerpunktabschnitt* as the lateral width of the main attack. In this game, it means the number of battlefields the player wishes to concentrate resources. It is effectively a subjective

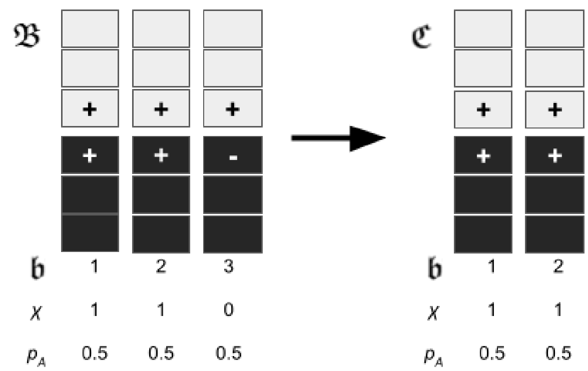



**Figure 4.2:** The delay of resolution in $\mathfrak{b}^3$ takes it out of the contest.

estimation of $|\mathfrak{C}|$. It affects how one would allocate troops across battlefields, which in turn, affects the probability of winning the game.

Suppose we have a game $\mathfrak{B} = \{\mathfrak{b}^1, ..., \mathfrak{b}^{10}\}$. $t = 70, w = 3$. Assuming $f_B$' is an evenly split strategy with no negative object ($f_B$' = $\{+7, +7, +7, +7, +7, +7, +7, +7, +7, +7\}$). A can choose one of the two options:

1. $f_A$ = $\{+21, +21, +21, -1, -1, -1, -1, -1, -1, -1\}$, by which A assumes $|\mathfrak{C}| = w_A$;
2. $f_A$ = $\{+13, +13, +13, +13, +13, -1, -1, -1, -1, -1\}$, by which A assumes $|\mathfrak{C}| = (w_A + w_B - 1)$.

These two strategies are fundamentally different, because they play different games:

- $f_A$1 is an "all-in" approach that takes the concept of *Schwerpunkt* to an extreme – a 3/3 game with higher $p_A$ for individual battlefields;
- $f_A$2 plays a 3+/5 game, with relatively lower $p_A$ for individual battlefields.

Below is a comparison between $f_A$1 and 2, as $|\mathfrak{C}|$ increases:

| $P_A$ \ $|\mathfrak{C}|$ / $f_A$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1. | 0.00 | 0.00 | 0.42 | 0.48 | 0.52 | 0.15 | 0.04 | >0.01 | >0.01 | >0.01 |
| 2. | 0.00 | 0.00 | 0.27 | 0.56 | 0.77 | 0.47 | 0.19 | 0.05 | 0.01 | >0.01 |
| $w$ | 3 | 3 | 3 | 3 | 3 | 4 | 5 | 6 | 7 | 8 |

**Table 4.1**

This tells us that it is preferable to focus on $|\mathfrak{C}|$ rather than $w_A$ battlefields. Note that practically, $|\mathfrak{C}|$ cannot exceed 5 without A attacking in the diversionary battlefields, but it shows us the complete trend of $P_A$: In either option, the probability of success reaches its maximum when $|\mathfrak{C}| = 5$, because the contest is decisive after *exactly* 5 resolutions. This is determined by $w_A$ and $w_B$, irrespective of either player's strategy.

Below is the analysis of the probabilities of different outcomes with The two response strategies.

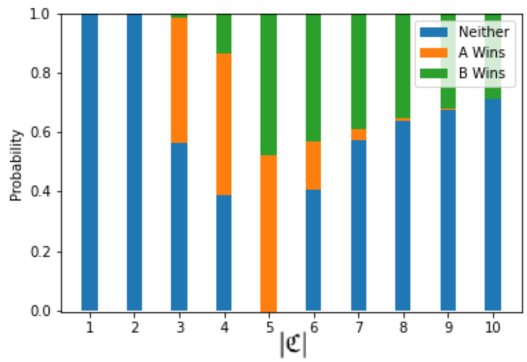



**Figure 4.3:** $f_A$1 is optimized for 3-battlefield contests

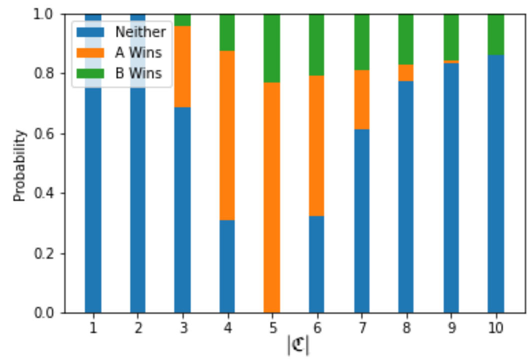



**Figure 4.4:** $f_A$2's performance begins to overtake $f_A$1 after 4 battlefields

As shown in Figures 4.3 and 4.4, indecision is entirely absent at the $(w_A + w_B - 1)$-th resolution, the equilibrium began immediately to restore itself as mutual destruction takes effect.

We can see how many positive objects $f_A$ has as a way of adjusting individual $p_A$'s in $\mathfrak{C}$. After 5 resolutions, $P_A 1 < P_A 2$, which would make the second strategy dominant. But if we check at the end of 3 resolutions, the probabilities are quite different:

$$P_1^{(3)} = 0.625^3 \approx 0.24$$
$$P_2^{(3)} = 0.5^3 \approx 0.13$$

This means $f_A$1 is better than $f_A$2 *before* the second strategy can catch up. Unfortunately, with three resolutions, the game is more likely to be in a state of indecision (draw) than A winning. It may be considered as a temporal advantage if over-concentration of sufficient resources allows $P_1 > 0.5$ before the $(w_A + w_B - 1)$-th resolution. But if we hold $t$ constant, and there is no time constraints, $f_A$2 is always preferable to $f_A$1, because over-concentration increases the risk by eliminating redundancy.

## § 4.3 Disposition and Probability

Having discovered the unique property of $(w_A + w_B - 1)$-battlefield contest, our problem can be re-phrased as "whether some battlefields in contest should have higher $p_A$'s than others, or they should be as close to each other as possible". In this test, only the contest is concerned. The contest strategies look like this

| $f$ \ $\lvert\mathfrak{C}\rvert$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| B' | +40 | +40 | +40 | +40 | +40 | +40 | +40 |
| A1 | +40 | +40 | +40 | +40 | +40 | +40 | +40 |
| A2 | +56 | +56 | +56 | +56 | +56 | -0 | -0 |
| A3 | +70 | +70 | +70 | +70 | -0 | -0 | -0 |

**Table 4.2**

This is a classical 4+/7 symmetrical majoritarian Blotto game, in which mirroring strategy is the Nash equilibrium. Strategies $f_A2$ and $f_A3$ transform it into different games: a 4+/5 game and a 4/4 game respectively. We notice that the fail-safe redundancy decreases as the extent of over-concentration grows. The figure below shows the probability of winning as area under curve:

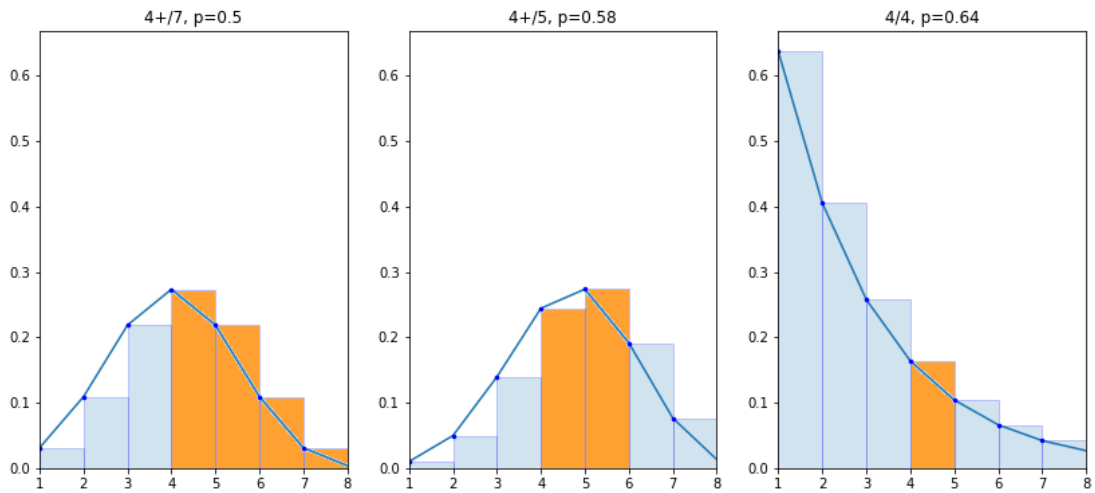



**Figure 4.5:** Comparison between different levels of concentration

The increase in individual battlefields does not offset the loss in highlighted areas. The practical implication is that there is no need to apply the *Schwerpunkt* principle recursively, because the price of strategic freedom is higher than the reward of improving the odds for individual battlefields.

§ 4.4 Symmetrical Game

Suppose the game is $t = 28, w = 2, m = 7$ We begin by evenly distribute troops across all battlefields. The *Schwerpunkt* is identified with the $(w_A + w_B - 1)$ battlefields that are the weakest.

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f_B{}'$ | $-4$ | $-4$ | $-4$ | $-4$ | $+4$ | $+4$ | $+4$ |
| $f_A$ | $-4$ | $-4$ | $-4$ | $-4$ | $+4$ | $+4$ | $+4$ |
| $\mathfrak{C}$ | $\{\mathfrak{b}^5, \mathfrak{b}^6, \mathfrak{b}^7\}$ | | | | | | |
| $\mathfrak{D}$ | $\{\mathfrak{b}^1, \mathfrak{b}^2, \mathfrak{b}^3, \mathfrak{b}^4\}$ | | | | | | |
| $P_A$ | 0.5 | | | | | | |

**Table 4.3:** Initial evenly split strategy

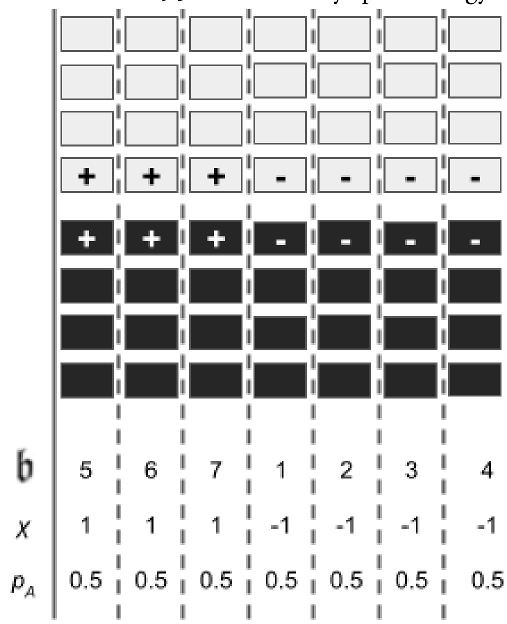



**Figure 4.6:** Arranged from the weakest to the strongest

From Figure 4.6, we can clearly see that A's troops in $\mathfrak{b}^{1,2,3,4}$ are not required, because B was not attacking. These battlefields are able to transfer their troops to $\mathfrak{b}^{5,6,7}$ to increase the odds.

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f_B{}'$ | $-4$ | $-4$ | $-4$ | $-4$ | $+4$ | $+4$ | $+4$ |
| $f_A$ | $-0$ | $-0$ | $-0$ | $-0$ | $+10$ | $+9$ | $+9$ |
| $\mathfrak{C}$ | $\{\mathfrak{b}^5, \mathfrak{b}^6, \mathfrak{b}^7\}$ | | | | | | |
| $\mathfrak{D}$ | $\{\mathfrak{b}^1, \mathfrak{b}^2, \mathfrak{b}^3, \mathfrak{b}^4\}$ | | | | | | |
| $P_A$ | $\approx 0.78$ | | | | | | |

**Table 4.4:** Initial evenly split strategy

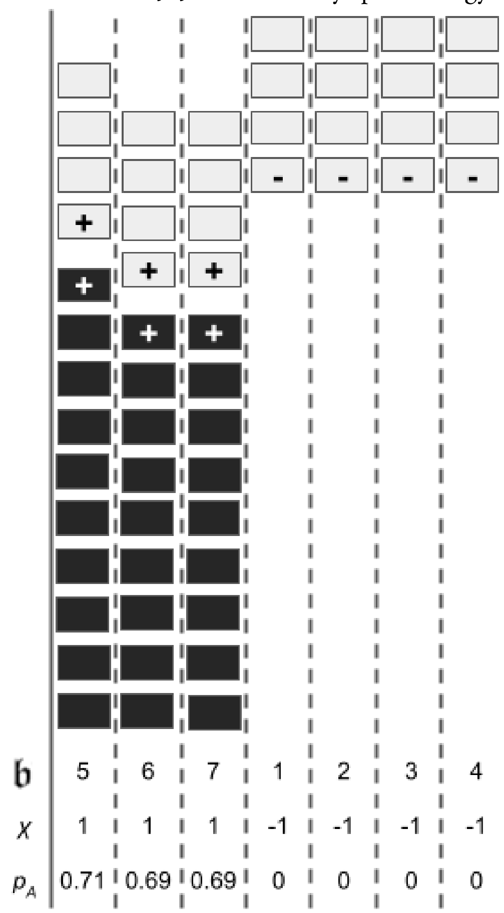



**Figure 4.7:** Oblique order with complete concentration of troops

This example also exploits the qualitative weakness of positive objectives mentioned in the previous chapter, used when no numerical advantage present itself. When comparing with the *prima facie* probability of 0.5 in a decisive contest, The strategy

$$f_A = \{-0, -0, -0, -0, +10, +9, +9\}$$

raises the probability to 0.78 by concentrating on the three decisive battlefields $\mathfrak{b}^{5,6,7}$. One should not forget that the goal is to seek out battlefields in which one has disproportionate advantage, not to maximize concentration for its own sake. If we continue the game from B's perspective, it will be apparent that the degree of concentration is no safe indicator of the effectiveness of strategy: Suppose B was given $f_A{}' = \{-0, -0, -0, -0, +10, +9, +9\}$. B would have to acknowledge the importance of the battlefields

$\mathfrak{b}^{5,6,7}$, but they are not decisive in B's strategy, because contesting in strong battlefields is inefficient. Instead, B should focus on the unopposed battlefields $\mathfrak{b}^{1,2,3}$:

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $f_A{}'$ | $-0$ | $-0$ | $-0$ | $-0$ | $+9$ | $+9$ | $+10$ |
| $f_B$ | $+9$ | $+8$ | $+8$ | $-0$ | $-1$ | $-1$ | $-1$ |
| $\chi$ | 1 | 1 | 1 | n/a | 0.82 | 0.8 | 0.8 |
| $p_B$ | 1 | 1 | 1 | 0 | 0.09 | 0.1 | 0.1 |
| $\mathfrak{C}$ | $\{\mathfrak{b}^1, \mathfrak{b}^2, \mathfrak{b}^3\}$ | | | | | | |
| $\mathfrak{D}$ | $\{\mathfrak{b}^4, \mathfrak{b}^5, \mathfrak{b}^6, \mathfrak{b}^7\}$ | | | | | | |
| $P_B$ | 1 | | | | | | |

**Table 4.5:** Note that $\mathfrak{b}^4$ is abandoned by both sides

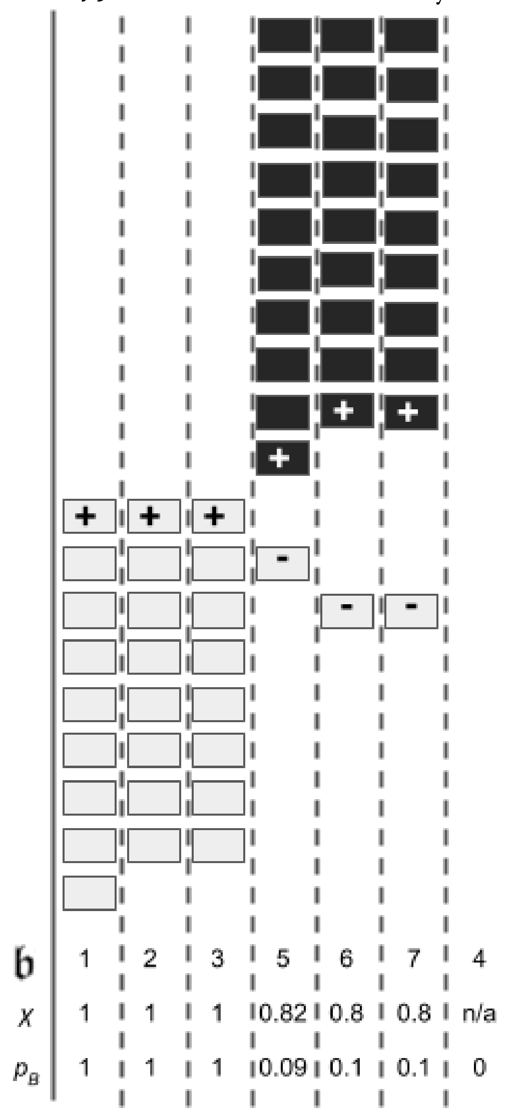



**Figure 4.8:** Concentration as a matter of principle

In this instance, B is guaranteed to win by avoiding the entirety of A's troops in battlefields $\mathfrak{b}^{1,2,3}$. Even though B's troops is more spread out than A's. A *Schwerpunkt*-based strategy, if not quickly realized, leaves itself vulnerable to exploitation by another *Schwerpunkt*-based strategy.

One can appreciate the tremendous leverage gained from picking the right targets from a glance at Figure 4.8: In fact, B can achieve this outcome with as few troops as 5, no more than the stones with which David slew the legendary Goliath.

## § 4.5 The End of Myth

In the beginning, we referenced the sizes of armies of both sides, and noted the disparity. This test looks for the connections between $P_A$ and $t_A$ when A is disadvantaged. In all three games, $t_A$ is gradually reduced while $t_B$ holds still, without changing the strategy of either player. In the first game, $w_A = w_B = 2$, and the starting conditions are as below:

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathfrak{f}_B$ | $-4$ | $-4$ | $-4$ | $-4$ | $+4$ | $+4$ | $+4$ |
| $\mathfrak{f}_A$ | $-0$ | $-0$ | $-0$ | $-0$ | $+10$ | $+9$ | $+9$ |

**Table 4.6**

The second game begins with $t_A = t_B = 280$.

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathfrak{f}_B$ | $-40$ | $-40$ | $-40$ | $-40$ | $+40$ | $+40$ | $+40$ |
| $\mathfrak{f}_A$ | $-0$ | $-0$ | $-0$ | $-0$ | $+94$ | $+93$ | $+93$ |

**Table 4.7**

The final game in this set is exactly the same condition as the previous one, except $w = 3$. It changes the troop disposition to cover $|\mathfrak{C}| = 5$, but the principle is the same:

| $\mathfrak{b}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\mathfrak{f}_B$ | $-40$ | $-40$ | $+40$ | $+40$ | $+40$ | $+40$ | $+40$ |
| $\mathfrak{f}_A$ | $-0$ | $-0$ | $+56$ | $+56$ | $+56$ | $+56$ | $+56$ |

**Table 4.8**

The results are as following:

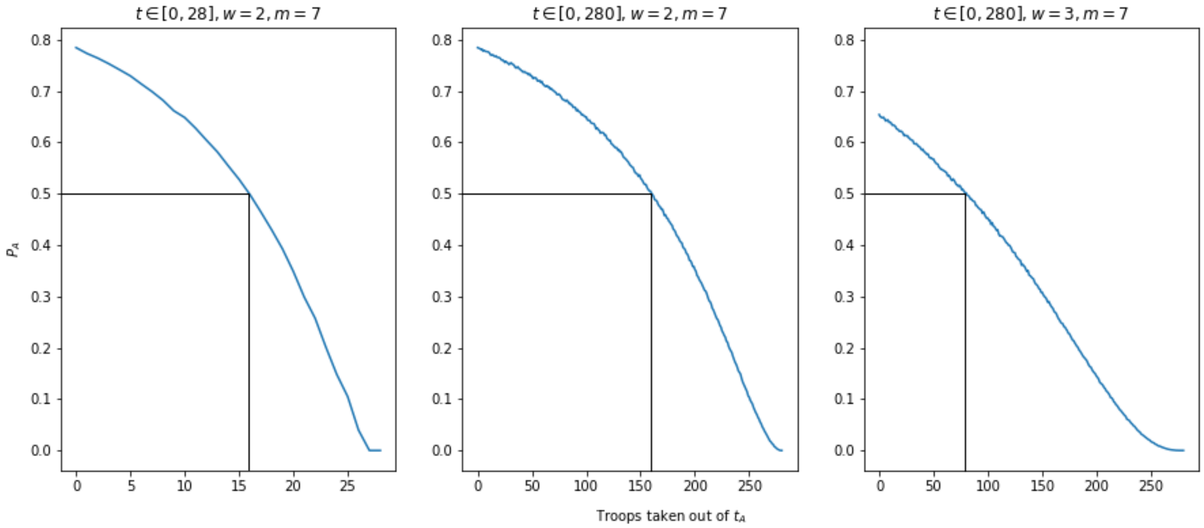



**Figure 4.9**

Assuming the strategy does not change, the trend of winning probability $P_A$ is not affected by the total number of troops: Although a large number might allow more minute adjustments as shown in tables 4.6 and 4.7 ({10,9,9} does not become {100,90,90}, but a more evenly distributed {94,93,93}), their improvements are marginal. When comparing the first two and the third game, the first two are 2+/3 games, the third is a 3+/5 game. If we consider $P_A = 0.5$ as the equilibrium, A can do away with over half of the total troops, and still remain on equal footing when A only has to win a 2+/3 game, the 3+/5 game shown in the last panel is much less generous. When $m$ stays the same, the increase in $|\mathfrak{C}|$ increases B's concentration without B's active adjustment in strategy.

Following up on the last game, the next is to explore the role of $m$. Both players retain their respective strategies (A still concentrate on 3 battlefields, while B evenly split troops across all $m$ battlefields).

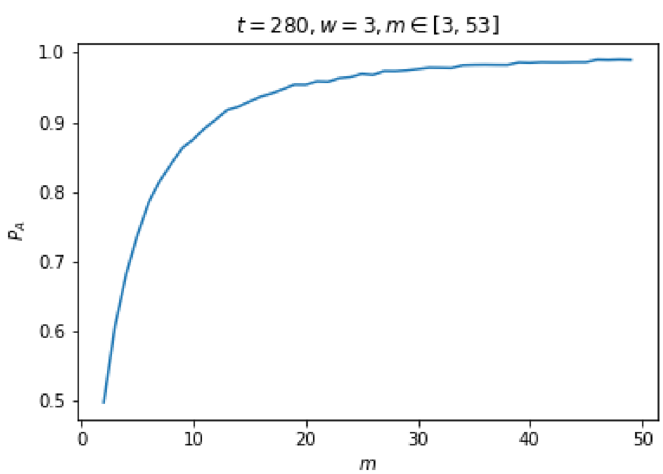



**Figure 4.10**

It should be noted that $f_B$ is an evenly split strategy, which represents the worst-case scenario. Below is a comparison with a randomly assigned strategy:

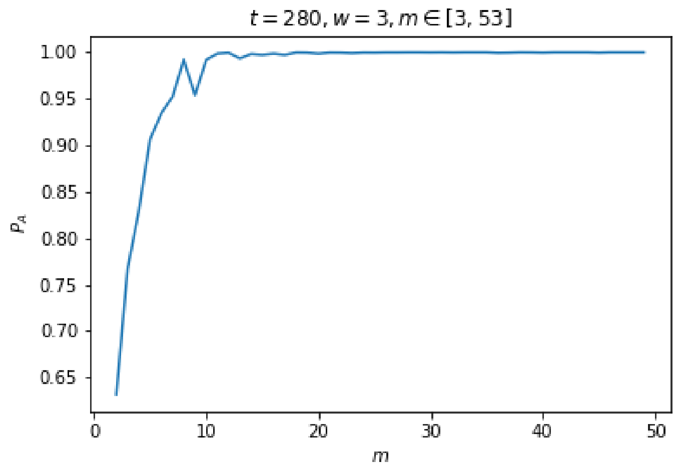



**Figure 4.11:** $f_B$ is random, note the scale on the y-axis is slightly higher

When B does not assign troops evenly, the diluting effect of $\widehat{\mu_B} = \frac{t_B}{m}$ is compounded by $\frac{\mu_A^x}{\mu_A^x + \mu_B^x}$, where $\mu_B^x < \widehat{\mu_B}$ : One can always find a weak spot when the opponent's strategy is not evenly distributed, which makes the series converge quicker on 1. In either case, the *Schwerpunkt* player benefits from greater diversion. This phenomenon fits the observation that an increase in the dimensions of the conflict is inherently beneficial to the weaker player (whose incentive in employing novel strategies is greater than their stronger opponent) [18].

## § 4.6 Troop Transfer

We noted that A was able to move more troops from $\mathfrak{D}$ to $\mathfrak{C}$, The restriction largely came from chronological ordering: When the improvement in $\chi^1$ allows it of offset the change in $\chi^2$, A is able to control the timing. This ability is more affected by $f_B$ than $t_A$ and $t_B$:

Suppose we have two battlefields $\mathfrak{b}^1 \in \mathfrak{C}$ and $\mathfrak{b}^2 \in \mathfrak{D}$. In either battlefield, $\widehat{\mu_A} = 150$, $\mu_B^1 = 100$, $\mu_B^2 = 200$. If we began to transfer troops $\mu_A^2$ to $\mu_A^1$, we can expect the marginal utility $\frac{1}{1+\mu_B}$, i.e. the amount by which $\chi$ is changed per transferred troop, to decrease in $\mathfrak{b}^1$ and increase in $\mathfrak{b}^2$, until A must stop in order to avoid subverting the preferred temporal relation $\chi^1 > \chi^2$.

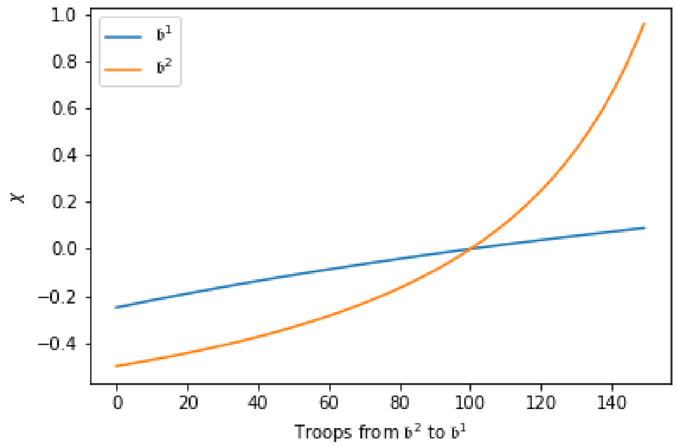



**Figure 4.12:** At some point $\chi^2$ will overtake $\chi^1$

This threshold is the maximum troops A can spare from the diversionary battlefield without abdicating the control of chronology. In another word, the maximum concentration that can be achieved in $\mathfrak{b}^1$($\mathfrak{b}^{c-}$ in general). We observe that $\mu_A$ can achieve certain leverage, sometimes with less number than $\mu_B$, although higher total number is still advantageous. But when this critical mass is not reached, A cannot formulate any meaningful concentration-based strategy. The critical mass of A is, to a large degree, affected by the B's own degree of concentration:

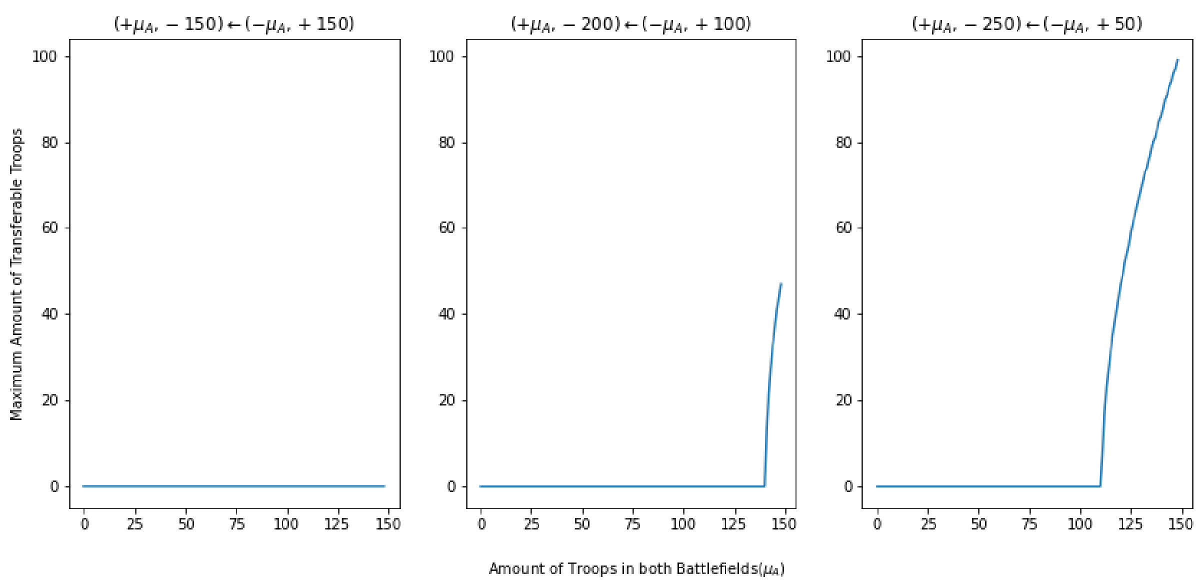



**Figure 4.13:** Depends on how B assigns troops and how many troops A has, the latter might be able to spare the majority of defensive troops or none at all.

The first graph describes an awkward situation like the one described in table 4.3, where A cannot afford take off a single troop without being the stronger player in the first place ($\{(+150, -150), (-150, +150)\}$). When $\mu$ manipulation avails nothing, one must turn attention towards $\kappa$.

Note that the results are only between two battlefields: If we look at the middle and the right graph, we see that A was able to spare troops when $\mu_A = 141$ and $\mu_A = 111$ respectively. If we only consider these two battlefields, it means A is capable of wresting away B's control of timeline at $t_A = 282$ and $t_A = 222$, even if B outnumbers A by $\%6$ and $\%26$ respectively. This advantage multiplies with number of battlefields: If there were two of each such battlefields, A would be able to attain the same strategic advantage with 564 and 444 troops against B's 600 troops.

To summarize the roles of the variables in the setup of the game, we can consider this thought experiment: If A was offered a chance to add 1 to either $w_B$, $m$ or $t_A$ before the game, $t_A + 1$ would be a slight bonus,$m + 1$ might give A a more noticeable boost if B did not concentrate, $w_B + 1$ would give A a significant advantage.

# CHAPTER 5

# CONCLUSION

This paper offers a framework to pair game-theoretical models with philosophy as a tool for resource allocation analysis, just as we did with the Epaminondas' troop deployment at the battle of Leuctra.

The Battle of Leuctra was the first recorded instance of a successful application of oblique order. Diodorus recorded that Epaminondas gathered his best warriors on one wing to be used in decisive action. The weakest of them were put on the other side, and bade them to avoid combat. (*Bibliotheca Historica* XV.55,[4]). The Boeotian attack on the Lacedaemonian right wing was successful. Xenophon reported that the the Boeotian formation was fifty ranks deep at the point, while the Lacedaemonians had twelve (*Hellenica* VI.4.12[24]). not only broke the formation, but also slew Cleombrotus, who was both the commander and the political leader of the faction. Abandoned by her allies, Lacedaemon was forced to abdicate hegemony of Greece. The Thebans erected a monument, part of it stands on the ancient battlefield to this day.

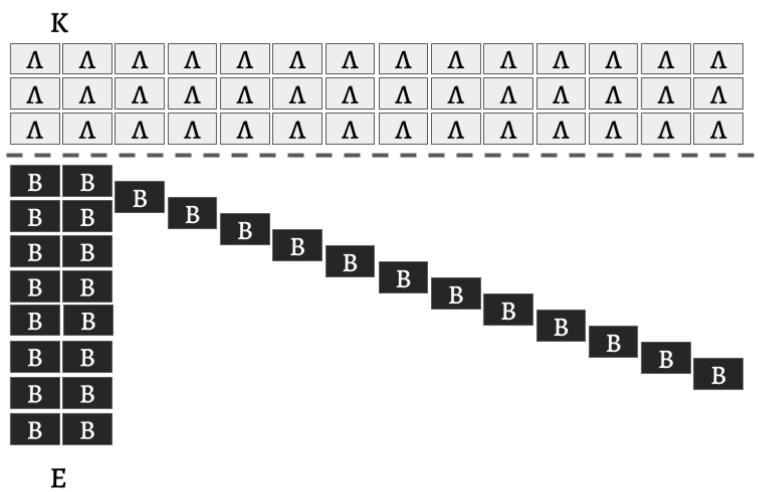



**Figure 5.1:** Epaminondas' oblique order at the Battle of Leuctra

There are several matters that, due to constraint of scope, were not addressed. I shall briefly discuss them below: First, much of the strategy relies heavily on the CSF as well as how the $\chi$ value is calculated. Many other CSF's exist, such as the difference-form CSF [12], and the weighted asymmetric lottery CSF [19]. Since the CSF is the crux of the game rules, the algorithm to generate strategies are likely particular to a specific CSF, although the principles that guided the design of algorithm likely retains generality. Since $\chi$ is a function of the CSF, it is also affected by the change. Furthermore, we have taken no account of the complications that might arise from large number of troops. For example, a battlefield $\mathfrak{b}^i = ((500, +1), (500, +1))$ would resolve faster than one with $\mathfrak{b}^j = ((5, +1), (5, -1))$. Second, all battlefields have the same payoffs in this game. It is possible to assign battlefields with different values $u$

and change the victory condition from $w$ to $u$. These additions are circumstantial, and must be tailored to specific context. Third, the partial observability ("fog", in Clausewitizian terms) of war is completely eliminated in our analysis. Of the few works on Blotto games with incomplete information, [1] examined Blotto game using auction CSF, and [11] used Blotto game to test bounded rational agents. An important reason that this simple principle is not followed is because of uncertainty: Looking at Table 4.4 from the previous chapter, if $f_B{}'$ was noisy, B would surprise A by attacking in the uncontested $\mathfrak{b}^{1,2,3,4}$. A single such surprise attack can nullify the advantage of *Schwerpunkt* by displacing $\mathfrak{b}^7$ in $\mathfrak{C}$, reducing $p_A$ from $0.78$ to $0.49$. In this case, A would have the incentive to spread out troops in a more evenly fashion to mitigate the damage of such surprise attacks, weakening the *Schwerpunkt* in the process.

Fourth, with respect to battle simulation, the current one-dimensional battlefield space can only model head-on confrontations, it cannot model more complex situations, such as the perpendicular order.

Finally, we should address the mythical fortune of the underdog: What we have shown creates a specific type of favorable conditions in a friction-less and completely observable environment within a defined scope. When these assumptions are challenged, the risks in such games become more opaque, and the inherent aggression in this strategy may, in the end, contribute more to miscalculation. Understanding how such gambits could work and the assumptions on which it relies is the first step to avoid reckless ventures.

APPENDIX

The *Schwerpunkt* Algorithm

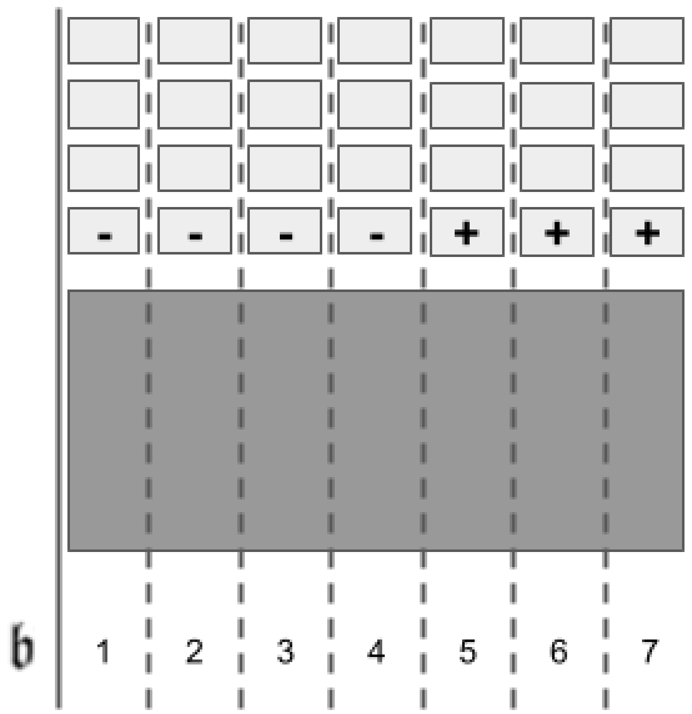



**Figure 5.2:** Begin, $w = 2$

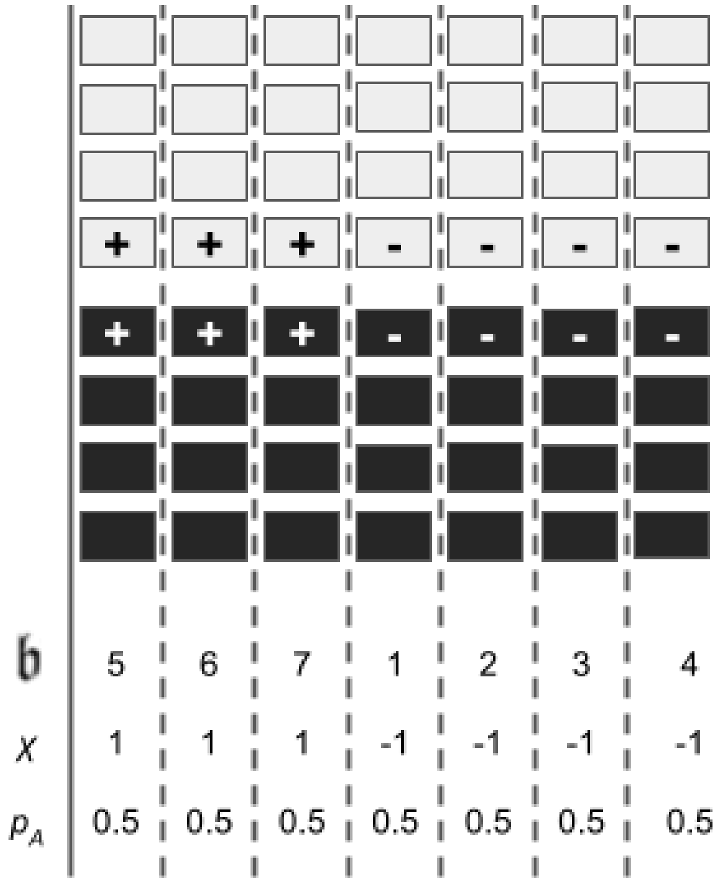



**Figure 5.3:** Assign troops evenly and find weakness

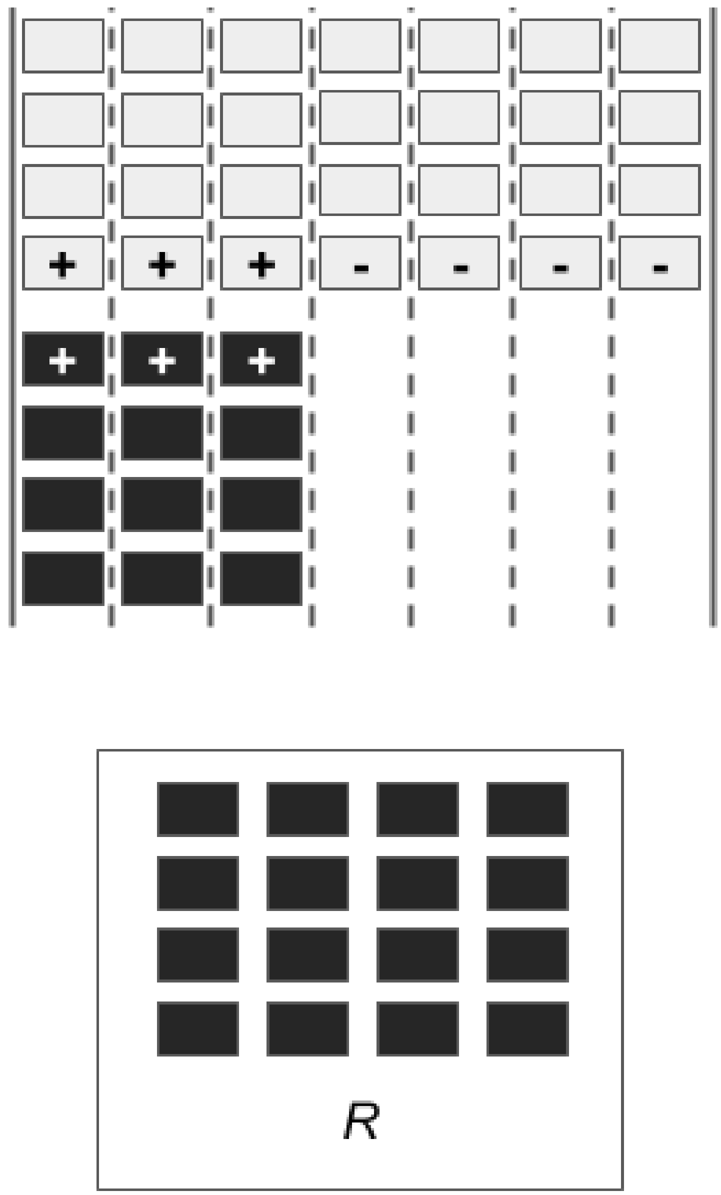



**Figure 5.4:** Taking off troops from diversionary battlefields to form a reserve $R$

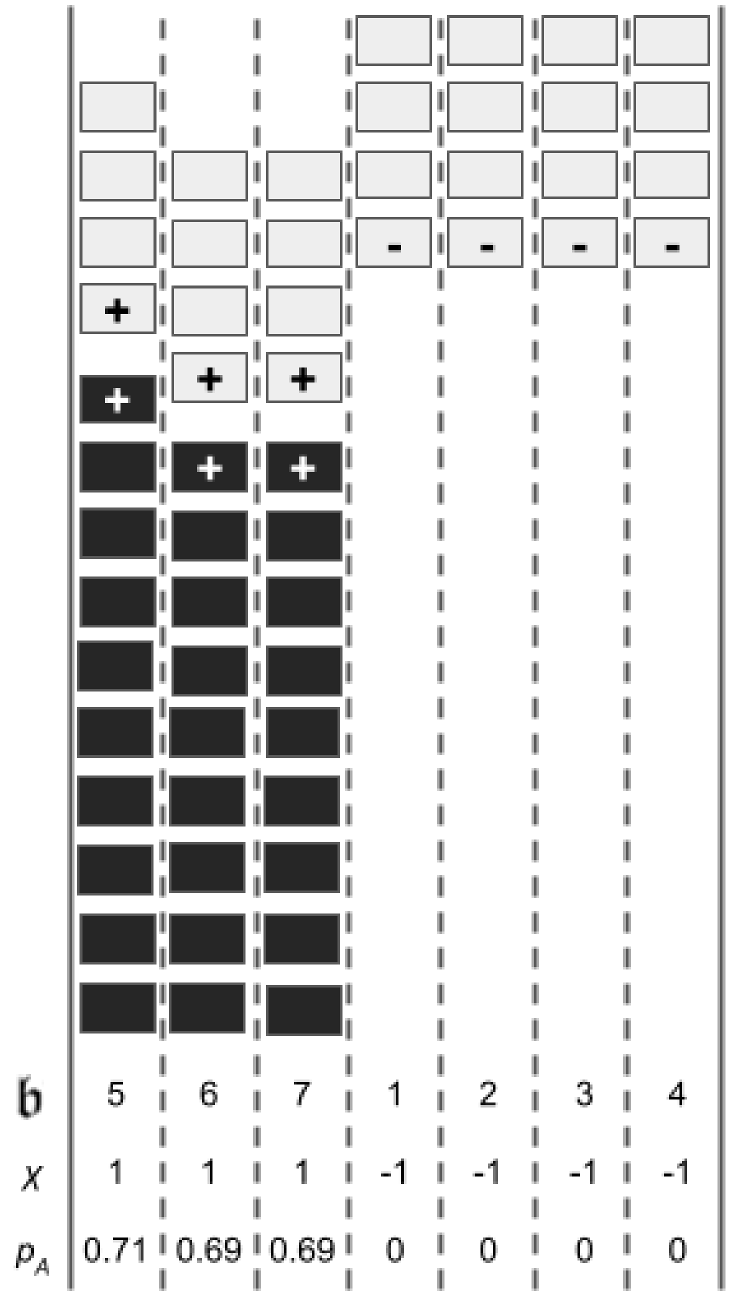



**Figure 5.5:** Putting $R$ in the contest battlefields

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